

ON THE RESOLUTION GRAPH OF A PLANE CURVE

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ABSTRACT. We show that the resolution graph of a plane curve singularity admits a canonical decomposition into elementary graphs.

1. INTRODUCTION

The main purpose of this paper is to give a canonical decomposition of the resolution graph of an arbitrary plane curve that generalizes the decomposition of the resolution graph of an irreducible plane curve introduced by Brieskorn and Knorrer remark in [1]. The elementary graphs of this decomposition are of two types: *smooth type* (all branches are smooth, see Example 4.4) and *cusp type* (all branches are "cusps" and the graph is linear, see Example 6.6).

One can find in [4] a closed form description for the local fundamental group of a plane curve, given the Puiseux expansions of the branches of the curve or its Eggers tree. One of the motivations of this paper is to clear the way to give a closed form description of the local fundamental group of a plane curve from its equations (or its resolution graph) that can be implemented in a computer ([2]). Yang Jingen gave in [6] an interesting characterization of the resolution graphs.

Section 2 recalls several topological invariants of plane curves and its behaviour by blow up. Theorem 3.6 shows that a normal crossings graph is a resolution graph if and only if it is contractible. Readers willing to accept this result may skip most of sections 2 and 3. Sections 4, 5 and 6 are dedicated to the characterization of the elementary graphs. Section 7 proves the main result.

2. INVARIANTS OF PLANE CURVE SINGULARITIES

Let Y be the germ of an irreducible plane curve defined by $f \in \mathbb{C}\{x, y\}$. Assume that the tangent cone of Y is transversal to $\{x = 0\}$ and Y has multiplicity k . By the Puiseux Theorem there are a positive integer n and complex numbers c_i , $i \geq n$, such that $(n, k) = 1$, $c_n \neq 0$ and Y admits the local parametrization

$$(1) \quad x = t^k, \quad y = \sum_{i \geq n} c_i t^i.$$

Consider the injective morphism from $\mathbb{C}\{x\}$ to $\mathbb{C}\{t\}$ that takes x into t^k . We will sometimes denote t by $x^{\frac{1}{k}}$. For $\alpha \in \mathbb{Q}_+$, set $a_\alpha = c_i$ if $\alpha = i/k$ and

$a_\alpha = 0$ if $k\alpha \notin \mathbb{Z}$. We say that (1) and $y = \sum_\alpha a_\alpha x^\alpha$ are *Puiseux expansions* of Y relatively to the system of local coordinates (x, y) .

There is a certain ambiguity on the determination of the coefficients of the Puiseux expansion of a plane curve. When we compare two Puiseux expansions

$$(2) \quad y = \sum_\alpha a_\alpha x^\alpha \quad \text{and} \quad y = \sum_\alpha b_\alpha x^\alpha,$$

we shall assume that $a_\alpha = b_\alpha$ for $\alpha \leq \beta$ if $\sum_{\alpha \leq \beta} a_\alpha x^\alpha$ and $\sum_{\alpha \leq \beta} b_\alpha x^\alpha$ are parametrizations of the same plane curve (see Section 4.1 of [5]).

If Y admits the Puiseux expansion $y = \sum_\alpha a_\alpha x^\alpha$, there are unique rational numbers $\alpha_1 < \dots < \alpha_g$ such that:

- (1) For $1 \leq k \leq g$, $a_{\alpha_k} \neq 0$.
- (2) Set $\alpha_{g+1} = +\infty$. If $a_\alpha \neq 0$ and $\alpha < \alpha_{k+1}$, α is a linear combination with integer coefficients of $\alpha_1, \dots, \alpha_k$.
- (3) α_i , $i \in \{2, \dots, g\}$, is not a linear combination with integer coefficients of $\alpha_1, \dots, \alpha_{i-1}$.

The rational numbers $\alpha_1, \dots, \alpha_g$ are called the *Puiseux exponents* of Y .

Given two plane curves Y, Z with Puiseux expansions (2), we call *contact exponent* of Y and Z to the smallest γ such that $a_\gamma \neq b_\gamma$. We will denote it by $o(Y, Z)$.

Let Y, Z be two plane curves with Puiseux expansions (2). Given a positive rational γ , $o(Y, Z) = \gamma$ if and only if $y = \sum_{\alpha \leq \beta} a_\alpha x^\alpha$ and $y = \sum_{\alpha \leq \beta} b_\alpha x^\alpha$ define the same curve for all positive rational $\beta < \gamma$ but $y = \sum_{\alpha \leq \gamma} a_\alpha x^\alpha$ and $y = \sum_{\alpha \leq \gamma} b_\alpha x^\alpha$ define different curves.

Notice that $o(Y, Z) = 1$ if and only if Y and Z have different tangent cones.

Theorem 2.1. *Let \tilde{Y} be the strict transform of an irreducible plane curve Y . Let E be the exceptional divisor of the blow up. Assume that Y has Puiseux exponents $\alpha_1, \dots, \alpha_g$.*

- (1) *If $\alpha_1 \geq 2$, \tilde{Y} is transversal to the exceptional divisor and has Puiseux exponents $\alpha_1 - 1, \alpha_2 - 1, \dots, \alpha_g - 1$.*
- (2) *Assume that $\alpha_1 < 2$. Then \tilde{Y} is tangent to the exceptional divisor and $o(\tilde{Y}, E) = 1/(\alpha_1 - 1)$.*
 - (a) *If $1/(\alpha_1 - 1)$ is not an integer, the Puiseux exponents of \tilde{Y} are $1/(\alpha_1 - 1), \alpha_2/(\alpha_1 - 1) - 1, \dots, \alpha_g/(\alpha_1 - 1) - 1$.*
 - (b) *If $1/(\alpha_1 - 1)$ is an integer, the Puiseux exponents of \tilde{Y} are $\alpha_2/(\alpha_1 - 1) - 1, \dots, \alpha_g/(\alpha_1 - 1) - 1$.*

Proof. It follows from Theorem 3.5.5 of [5]. □

Corollary 2.2. *Let X be a smooth complex surface. Let E be an exceptional divisor of X . Let $\pi : X \rightarrow \underline{X}$ be the blow down of X along E . Let Y be the germ of a irreducible plane curve at a point σ of E . Assume that Y has Puiseux exponents $\alpha_1, \dots, \alpha_g$.*

- (1) If Y is transversal to E , $\pi(Y)$ has Puiseux exponents $\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_g + 1$.
- (2) Assume that Y is tangent to E .
 - (a) If $o(Y, E)$ is not an integer, $o(Y, E) = \alpha_1$ and $\pi(Y)$ has Puiseux exponents $1 + 1/\alpha_1, (\alpha_2 + 1)/\alpha_1, \dots, (\alpha_g + 1)/\alpha_1$.
 - (b) If $o(Y, E)$ is an integer, $\pi(Y)$ has Puiseux exponents $1 + 1/o(Y, E), (\alpha_1 + 1)/o(Y, E), \dots, (\alpha_g + 1)/o(Y, E)$.

Theorem 2.3. Let Y, Z be irreducible plane curves with the same tangent cone. Let \tilde{Y}, \tilde{Z} be their strict transforms and E the exceptional divisor. Let $\mu [\nu, \tilde{\mu}, \tilde{\nu}]$ be the first Puiseux exponent of $Y [Z, \tilde{Y}, \tilde{Z}]$. Assume that $\nu \leq \mu$.

- (1) If $\nu \geq 2$, $o(\tilde{Y}, \tilde{Z}) = o(Y, Z) - 1$ and \tilde{Y}, \tilde{Z} are transversal to E .
- (2) If $\nu < 2$ and $\mu > 2$, $o(\tilde{Y}, \tilde{Z}) = 1$ and \tilde{Z} is tangent to E .
- (3) Assume that $\mu < 2$. Then \tilde{Y}, \tilde{Z} are tangent to E . Moreover, $\tilde{\mu} \leq \tilde{\nu}$.
 - (a) Assume that $o(Y, Z) > \mu$. Then $\mu = \nu$, $\tilde{\mu} = \tilde{\nu}$ and $o(\tilde{Y}, E) = o(\tilde{Z}, E) = \tilde{\mu} < o(Y, Z)/(\mu - 1) - 1 = o(\tilde{Y}, \tilde{Z})$.
 - (b) Assume that $o(Y, Z) = \mu$. Then $\mu = \nu$ and $o(\tilde{Y}, E) = o(\tilde{Z}, E) = \tilde{\mu} = \tilde{\nu} = o(\tilde{Y}, \tilde{Z})$.
 - (c) Assume that $o(Y, Z) < \mu$. Then $o(Y, Z) = \nu$ and $o(\tilde{Y}, E) = o(\tilde{Z}, E) = \tilde{\mu} < \tilde{\nu} = o(\tilde{Z}, E)$.

Proof. Statements (1) and (2) of Theorem 2.1 imply statement (2). Let (1) be a parametrization of Y . Therefore the curve \tilde{Y} has a parametrization given by

$$(3) \quad x = t^k, \quad \frac{y}{x} = \sum_{i \geq n} c_i t^{i-k} = t^{n-k} \gamma(t), \quad \gamma(0) \neq 0.$$

Assume $\nu \geq 2$. Then the parametrization (3) is a Puiseux expansion and statement (1) holds. Assume $\mu < 2$. In this case, the parametrization (3) is not a Puiseux expansion. Let $x = t^k$, $y = t^n \bar{\gamma}(t)$, $\bar{\gamma}(0) \neq 0$ be a Puiseux expansion of Z . Let $\alpha(t) = \sum_{i \geq 0} \alpha_i t^i$, $\bar{\alpha}(t) = \sum_{i \geq 0} \bar{\alpha}_i t^i$ such that $\alpha(t)^{n-k} = \gamma(t)$, $\bar{\alpha}(t)^{n-k} = \bar{\gamma}(t)$. Let $\beta(t) = \sum_{i \geq 0} \beta_i t^i$, $\bar{\beta}(t) = \sum_{i \geq 0} \bar{\beta}_i t^i$, such that $t = w\beta(w)$ is a solution of $w = t\alpha(t)$ and $t = w\bar{\beta}(w)$ is a solution of $w = t\bar{\alpha}(t)$. To prove statement (3) it suffices to show that if $\alpha(t) \equiv \bar{\alpha}(t) \pmod{(t)^i}$, $i \in \mathbb{N}$, $\beta(t) \equiv \bar{\beta}(t) \pmod{(t)^i}$. Also, if $\alpha_{i+1} \neq \bar{\alpha}_{i+1}$, $\beta_{i+1} \neq \bar{\beta}_{i+1}$. We can check easily that α_i depends only on c_n, \dots, c_{n+i} , for all $i \geq 0$, and $\alpha(0) \neq 0$. From the equality $w = t\alpha(t)$ we get

$$(4) \quad w = \sum_{i \geq 0} \sum_{j_0, \dots, j_i} \alpha_i \beta_{j_0} \dots \beta_{j_i} w^{j_0 + \dots + j_i + i + 1}.$$

From equality (4) we deduce the equations

$$(5) \quad \alpha_0 \beta_0 = 1,$$

$$(6) \quad \alpha_0 \beta_{k-1} + \sum_{i=1}^{k-1} \sum_{j_0+\dots+j_i=k-1-i} \alpha_i \beta_{j_0} \dots \beta_{j_i} = 0, \quad k \in \mathbb{N}.$$

Therefore $\beta(0) \neq 0$ and the parametrization

$$x = (w\beta(w))^{\frac{k}{(k,n-k)}}, \quad \frac{y}{x} = w^{\frac{n-k}{(k,n-k)}}$$

is a Puiseux expansion. Also, by induction, we prove that β_i depends only on $\alpha_0, \dots, \alpha_i$, for all $i \geq 0$. \square

Corollary 2.4. *Let X be a smooth complex surface. Let E be an exceptional divisor of X . Let $\pi : X \rightarrow \underline{X}$ be the blow down of X along E . Let Y, Z be two germs of irreducible plane curves at a point σ of E . The curves $\pi(Y), \pi(Z)$ are germs of irreducible plane curves at the point $\sigma(E)$ of \underline{X} , with the same tangent cone. Let $\mu [\nu]$ be the first Puiseux exponent of Y [Z]. Assume that $\mu \leq \nu$.*

- (1) *If Y, Z are transversal to E , $o(\pi(Y), \pi(Z)) = o(Y, Z) + 1$.*
- (2) *If Y is transversal to E and Z is tangent to E , $o(\pi(Y), \pi(Z)) = 1 + 1/\nu$.*
- (3) *Assume that Y and Z are tangent to E .*
 - (a) *If $o(Y, E) = o(Z, E) < o(Y, Z)$, $o(\pi(Y), \pi(Z)) = (\mu-1)(o(Y, Z)+1)$.*
 - (b) *If $o(Y, E) = o(Z, E) = o(Y, Z)$, $o(\pi(Y), \pi(Z)) = 1 + 1/o(Y, Z)$.*
 - (c) *If $o(Y, E) = o(Y, Z) < o(Z, E)$, $o(\pi(Y), \pi(Z)) = 1 + 1/o(Y, Z)$.*

3. THE COMBINATORICS OF THE RESOLUTION ALGORITHM

3.1. Let Y be the germ at the origin of a plane curve defined by $f \in \mathbb{C}\{x, y\}$. Let X be a polydisc where f converges. We can assume that Y has no singular points on $X \setminus \{(0, 0)\}$. Set $X_0 = X$, $Y_0 = Y$, $\sigma_0 = (0, 0)$. Let $\pi_1 : X_1 \rightarrow X_0$ be the blow up of X with center σ_0 . Set $D_1 = \pi_1^{-1}(\sigma_0)$. Let Y_1 be the strict transform of Y .

Let us define recursively a sequence of triples $(X_\ell, D_\ell, Y^{(\ell)})$, where X_ℓ is a smooth complex surface, D_ℓ is a divisor with normal crossings of X_ℓ and $Y^{(\ell)}$ is a union of germs of plane curves at points of D_ℓ . Let W_ℓ be set of points of $D_\ell \cap Y_\ell$ where $D_\ell \cup Y^{(\ell)}$ is not a divisor with normal crossings. Let $\pi_{\ell+1} : X_{\ell+1} \rightarrow X_\ell$ be the blow up of X_ℓ with center C_ℓ . Let $Y^{(\ell+1)}$ be the strict transform of $Y^{(\ell)}$ by $\pi_{\ell+1}$. Set $W_{\ell+1} = \pi^{-1}(D_\ell)$. After a finite number of steps $D_\ell \cup Y^{(\ell)}$ is a normal crossings divisor, W_ℓ is the empty set and $X_{\ell+1} = X_\ell$.

Let us describe this procedure in purely combinatorial terms.

Definition 3.1. *Consider the following data:*

- (1) *A weighted tree \mathcal{D} . The vertices of \mathcal{D} are called divisors. The weight ω_E of a divisor E of \mathcal{D} is a negative integer. We call ω_E the self-intersection number of E . If two divisors are connected by an edge, we say that they intersect each other.*

- (2) A nonempty set $\mathcal{o}_{\mathcal{D}}$. The elements of $\mathcal{o}_{\mathcal{D}}$ are called points of \mathcal{D} . If \mathcal{D} is the empty graph, $\mathcal{o}_{\mathcal{D}}$ has only one element. We associate to each point σ of \mathcal{D} a set of divisors \mathcal{D}_{σ} of \mathcal{D} . If $E \in \mathcal{D}_{\sigma}$, we say that σ belongs to E and E contains σ .
- (3) A family of disjoint finite sets \mathcal{C}_{σ} , where σ is a point of \mathcal{D} . The elements of \mathcal{C}_{σ} are called branches. We associate to each branch i of \mathcal{C}_{σ} a nonnegative integer g_i and rational numbers $\varepsilon_{i,j}$, $1 \leq j \leq g_i$. We call the rational numbers $\varepsilon_{i,j}$ the Puiseux exponents of the branch i . Let $\tilde{\mathcal{C}}_{\sigma}$ be the union of \mathcal{C}_{σ} with the set of divisors of \mathcal{D} that contain σ . We define a map $o : \{(i, j) : i, j \in \tilde{\mathcal{C}}_{\sigma}, i \neq j\} \rightarrow \mathbb{Q}$. We call $o(i, j)$ the contact order of i and j .

We call the set of data (1),(2),(3) a resolution step if it verifies the conditions:

- (1) A point of \mathcal{D} belongs to at most two divisors. Given two divisors, there is at most one point that belongs to both.
- (2) If a point of \mathcal{D} belongs to two divisors, they intersect each other.
- (3) If the graph \mathcal{D} is nonempty, each point of \mathcal{D} belongs to some divisor.
- (4) $o(i, j) \geq 1$ for each $i, j \in \tilde{\mathcal{C}}_{\sigma}$.
- (5) $o(i, j) = o(j, i)$ for each $i, j \in \tilde{\mathcal{C}}_{\sigma}$.
- (6) $o(i, k) \geq \min\{o(i, j), o(j, k)\}$ for each $i, j, k \in \tilde{\mathcal{C}}_{\sigma}$.
- (7) $1 = \varepsilon_{i,0} < \varepsilon_{i,1} < \dots < \varepsilon_{i,g_i}$ for each $i \in \mathcal{C}_{\sigma}$.
- (8) $\varepsilon_{i,j}$ is not a linear combination with integer coefficients of $\varepsilon_{i,1}, \dots, \varepsilon_{i,j-1}$ for $1 \leq j \leq g_i$.

We say that a resolution step is a plane curve if the graph \mathcal{D} is the empty graph. We say that a resolution step has normal crossings at a point σ of \mathcal{D} if the point σ belongs to exactly one divisor E , the point σ has exactly one branch Y and $o(Y, E) = 1$. We say that a resolution step is a normal crossings graph if it is normal crossings at each point σ of $\mathcal{o}_{\mathcal{D}}$. We will represent a normal crossings graph in the following way. We will connect by an edge the asterisc that represents a branch of Y to the vertex that represents a divisor if there is a point that belongs to both.

Let σ be a point of \mathcal{D} . Consider in \mathcal{C}_{σ} the equivalence relation $i \sim j$ if $o(i, j) > 1$. We call an equivalence class ℓ of \mathcal{C}_{σ} a tangent line of σ .

Let X be a smooth complex surface. Let D be a normal crossings divisor of X . We will call *divisors* of D to the irreducible components of D . Let Y be the union of a finite set of germs of plane curves of X at points of D . If D is not empty, $Y = \cup_{\sigma \in A} Y_{\sigma}$, where A is a finite subset of D and Y_{σ} is the germ at σ of a plane curve with no irreducible component contained in D . If D is empty, Y is the germ of a plane curve at a point σ_0 of X . Let us associate to a triple (X, D, Y) a resolution step denoted by $[X, D, Y]$. Let \mathcal{D} be the graph with vertices the divisors of D . Two vertices of \mathcal{D} are connected if they intersect each other. We will label each divisor E of D with its self-intersection number ω_E (see [7]). If D is not empty, set $\mathcal{o}_{\mathcal{D}} = A$.

Otherwise set $o_{\mathcal{D}} = \{\sigma_0\}$. Let \mathcal{D}_{σ} be the set of divisors of D that contain σ , for each $\sigma \in o_{\mathcal{D}}$. Let $Y_i, i \in \mathcal{C}_{\sigma}$, be the irreducible components of Y_{σ} . Let $\varepsilon_{ij}, 1 \leq j \leq g_i$, be the Puiseux exponents of Y_i for each $i \in \mathcal{C}_{\sigma}$. If $i, j \in \mathcal{C}_{\sigma}$ and $i \neq j$, set $o(i, j) = o(Y_i, Y_j)$. If $i \in \mathcal{C}_{\sigma}$ and $E \in \mathcal{D}_{\sigma}$, set $o(i, E) = o(Y_i, E)$. If $E, F \in \mathcal{D}_{\sigma}$ and $E \neq F$, set $o(E, F) = 1$.

Notice that $[X, D, Y]$ is normal crossings if and only if $D \cup Y$ is a normal crossings divisor. Moreover, $[X, D, Y]$ is a *first step* if and only if D is empty.

Definition 3.2. Let us define the blow up $(\overline{\mathcal{D}}, (\overline{\mathcal{C}}_{\sigma}))$ of a resolution step $(\mathcal{D}, (\mathcal{C}_{\sigma}))$. Let $c_{\mathcal{D}}$ be the set of $\sigma \in o_{\mathcal{D}}$ such that $(\mathcal{D}, (\mathcal{C}_{\sigma}))$ is not normal crossings at σ . The vertices of $\overline{\mathcal{D}}$ will be the vertices of \mathcal{D} and the elements of $c_{\mathcal{D}}$. For each divisor E of $\overline{\mathcal{D}} \setminus c_{\mathcal{D}}$ we set $\overline{\omega}_E = \omega_E - k$, where k is the number of elements of $c_{\mathcal{D}}$ that belong to E . We set $\overline{\omega}_E = -1$ for each element E of $c_{\mathcal{D}}$. Let $\sigma \in c_{\mathcal{D}}$. If σ belongs to a divisor E , we connect σ to E by an edge. If σ belongs to a pair of divisors E, F , we withdraw the edge that connects E to F and set

$$o_{\overline{\mathcal{D}}} = (o_{\mathcal{D}} \setminus c_{\mathcal{D}}) \cup \bigcup_{\sigma \in c_{\mathcal{D}}} \{ \text{tangent lines of } \sigma \}.$$

We set $\overline{\mathcal{C}}_{\sigma} = \mathcal{C}_{\sigma}$ for each $\sigma \in o_{\mathcal{D}} \setminus c_{\mathcal{D}}$. We set $\overline{\mathcal{C}}_{\ell} = \ell$ for each tangent line ℓ of $\sigma \in c_{\mathcal{D}}$. We define the Puiseux exponents and the contact exponents of $(\overline{\mathcal{D}}, (\overline{\mathcal{C}}))$ according to Theorems 2.1 and 2.3.

Definition 3.3. Let us define the blow down $(\underline{\mathcal{D}}, (\underline{\mathcal{C}}_{\sigma}))$ of the resolution step $(\mathcal{D}, (\mathcal{C}_{\sigma}))$. The vertices of $\underline{\mathcal{D}}$ are the vertices E of \mathcal{D} such that $\omega_E \neq -1$. For each vertex E of $\underline{\mathcal{D}}$ we set $\underline{\omega}_E = \omega_E + k$, where k is the number of divisors $F \in \mathcal{D} \setminus \underline{\mathcal{D}}$ such that F intersects E . If two divisors E, F of $\underline{\mathcal{D}}$ intersect a divisor of weight -1 of \mathcal{D} , we connect E and F by an edge. Let $o_{\mathcal{D}}^*$ be the set of points $\sigma \in o_{\mathcal{D}}$ such that σ does not belong to a divisor of weight -1 . We set $o_{\underline{\mathcal{D}}} = o_{\mathcal{D}}^* \cup (\mathcal{D} \setminus \underline{\mathcal{D}})$. If $\sigma \in o_{\mathcal{D}}^*$, set $\underline{\mathcal{C}}_{\sigma} = \mathcal{C}_{\sigma}$. If $E \in \mathcal{D} \setminus \underline{\mathcal{D}}$, i.e. $\omega_E = -1$, set $\underline{\mathcal{C}}_E = \bigcup \{\mathcal{C}_{\sigma} : \sigma \text{ belongs to } E\}$. We define the Puiseux exponents and the contact exponents of $(\underline{\mathcal{D}}, (\underline{\mathcal{C}}_{\sigma}))$ according to Corollaries 2.2 and 2.4.

The following result is an immediate consequence of the previous definitions.

Lemma 3.4. The blow up of the resolution step of (X, D, Y) equals the resolution step of the blow up of (X, D, Y) . The blow down of the resolution step of (X, D, Y) equals the resolution step of the blow down of (X, D, Y) .

Definition 3.5. We say that a normal crossings graph is a resolution graph if there is a plane curve that is transformed into the normal crossings graph by a sequence of blow ups. We say that a normal crossings graph is contractible if there is a sequence of blow downs that takes the normal crossings graph into a plane curve. On the step previous to the last blow down, we get the empty graph and the normal crossings graph reduces to a single vertex. We call this vertex the root of the normal crossings graph.

The root of a resolution graph is therefore the divisor created by the first blow up. We call *root of a resolution step* to the strict transform of the divisor created by the first blow up.

Theorem 3.6. *A normal crossings graph is a resolution graph if and only if it is contractible.*

Proof. If the normal crossings graph is contractible, we can blow it down into a plane curve and reverse the procedure, obtaining the initial resolution graph. \square

A resolution graph \mathcal{R} is *minimal* if there is no divisor E of weight -1 of \mathcal{R} such that after blowing down E we obtain a normal crossings graph. Given the germ of a plane curve Y , there is one and only one minimal resolution graph \mathcal{R}_Y that is a resolution graph of Y .

Given a divisor E of a resolution step we call *valence* of E to the number of divisors of that resolution step that intersect E . We say that a resolution step is *linear* if all of its divisors have valence 0, 1 or 2. If a resolution step has only one divisor, this divisor is simultaneously the root and a terminal vertex of the resolution step. Otherwise we call *terminal vertex* of a resolution step to its vertices of valence 1 that are not the root.

4. RESOLUTION GRAPHS OF SMOOTH TYPE

Let $Y [\Sigma]$ be a [smooth] plane curve of (X, o) . We call the pair (Y, Σ) a *logarithmic plane curve* if Y and Σ have no common irreducible components. We say that a logarithmic plane curve is *singular* if the curve $Y \cup \Sigma$ is not a normal crossings divisor. We call *branches* of (Y, Σ) to the branches of Y . Let C be a smooth curve transversal to Σ . Define $\tau_C(Y)$ as the smallest positive integer τ such that $o(Y_i, C) < \tau$ for each singular branch Y_i of Y .

Let $\pi : \tilde{X} \rightarrow X$ be the composition of blow ups that desingularizes $Y \cup \Sigma$. Let $\tilde{Y} [\tilde{\Sigma}]$ be the strict transform of $Y [\Sigma]$ by π . We call *resolution graph* of (Y, Σ) to the graph \mathcal{R}_Y^Σ obtained in the following manner: the vertices of \mathcal{R}_Y^Σ are the vertices of $\mathcal{R}_{Y \cup \Sigma}$ plus $\tilde{\Sigma}$; the branches of \tilde{Y} are the branches of $\widetilde{Y \cup \Sigma}$ minus $\tilde{\Sigma}$. We call $\tilde{\Sigma}$ the *root* of \mathcal{R}_Y^Σ . We set $\omega_{\tilde{\Sigma}} = -c - 1$, where c is the number of times we blow up a point of a strict transform of Σ .

We call *log resolution graph* to a pair $\mathcal{R}^\Sigma = (\mathcal{R}, \Sigma)$, where:

- (i) \mathcal{R} is a normal crossings graph;
- (ii) Σ is a divisor of valence 1 of \mathcal{R} ;
- (iii) If we consider Σ a branch of \mathcal{R} , \mathcal{R} becomes the resolution graph of a plane curve;
- (iv) The image $\underline{\Sigma}$ of Σ , by the blow down of \mathcal{R} into a plane curve, is a smooth component of Y ;
- (v) $\omega_\Sigma = -c - 1$, where c is the number of times necessary to blow down a divisor intersecting Σ when we blow down \mathcal{R} into a plane curve.

A normal crossings graph \mathcal{R} is a log resolution graph if and only if there is a logarithmic plane curve (Y, Σ) such that $\mathcal{R} = \mathcal{R}_Y^\Sigma$.

Let \mathcal{D} be a resolution step and E a smooth divisor of \mathcal{D} with weight ω_E and valence ϑ_E . We call *rectified weight* of E to the number $\omega_E + \vartheta_E$.

Definition 4.1. Let $\mathcal{R}[\mathcal{R}^\Sigma]$ be a $[\log]$ resolution graph. We say that $\mathcal{R}[\mathcal{R}^\Sigma]$ is a $[\log]$ elementary graph of smooth type if the root of $\mathcal{R}[\mathcal{R}^\Sigma]$ has rectified weight -1 and the other divisors of $\mathcal{R}[\mathcal{R}^\Sigma]$ have rectified weight 0 .

Theorem 4.2. Let (Y, Σ) be a logarithmic plane curve. Assume Σ smooth. The graph \mathcal{R}_Y^Σ is a log elementary graph of smooth type if and only if all branches of Y are non singular and transversal to Σ .

Proof. We will use the notations of Paragraph 3.1. Set $\Sigma^{(0)} = \Sigma$. Let $\Sigma^{(\ell+1)}$ be the strict transform of $\Sigma^{(\ell)}$ by $\pi_{\ell+1}$, for each $\ell \geq 0$. Assume that all branches of Y are non singular and transversal to Σ . Set $D_1 = \pi_1^{-1}(\Sigma)$. Consider the statements:

- (1) $_\ell$ $D_\ell \cup Y^{(\ell)}$ is normal crossings at each point of a non terminal divisor E .
- (2) $_\ell$ $\Sigma^{(\ell)}$ has rectified weight -1 and the remaining irreducible components of D_ℓ have rectified weight 0 .
- (3) $_\ell$ Each branch of $Y^{(\ell)}$ is non singular, intersects only one divisor and is transversal to this divisor.

Statements $(i)_\ell$, $i = 1, 2, 3$, $\ell \geq 1$, can be proved by induction in ℓ . Let N be the maximal element of $\{o(Y_i, Y_j) : i, j \in I, i \neq j\}$. The curve Y is desingularized after N blow ups. By $(2)_N$, \mathcal{R}_Y^Σ is a log elementary graph of smooth type.

Let us prove the converse. We associate to each vertex E of a rooted tree with root Σ its *depth*, the number of edges we have to cross in order to go from E to Σ . We call depth of the tree to the supreme of the depths of its elements. Let N be the depth of \mathcal{R}_Y^Σ . Let $\pi : \tilde{X} \rightarrow X$ be the minimal sequence of blow ups that desingularizes (Y, Σ) . Set $D_N = \pi^{-1}(\Sigma)$. Let $Y^{(N)}[\Sigma^{(N)}]$ be the strict transform of $Y[\Sigma]$ by π . Hence $Y^{(N)}, \Sigma^{(N)}, D_N$ verify $(i)_N$, $1 \leq i \leq 3$. The divisors of depth N have valence 1, hence weight -1 . Let us blow down these divisors. Let $Y^{(N-1)}, \Sigma^{(N-1)}, D_{N-1}$ be the images of $Y^{(N)}, \Sigma^{(N)}, D_N$ by the blow down. Statements $(1)_{N-1}, (2)_{N-1}$ are easily verified. If a branch of $Y^{(N-1)}$ was singular, its strict transform would be singular or tangent to a divisor. If a branch of $Y^{(N-1)}$ intersected two divisors, the blow up of its intersection would create a divisor of positive rectified weight. If a branch of $Y^{(N-1)}$ was tangent to a divisor, its strict transform would intersect two divisors. Hence we can iterate the procedure. By $(3)_0$ the branches of Y are smooth and transversal to Σ . \square

Corollary 4.3. Let Y be a singular plane curve. Its resolution graph \mathcal{R}_Y is an elementary graph of smooth type if and only if all branches of Y are non singular.

Proof. Let Y be a singular plane curve. Let Σ be a smooth curve transversal to Y . The graph \mathcal{R}_Y is an elementary graph of smooth type if and only if \mathcal{R}_Y^Σ is a log elementary graph of smooth type. \square

We shall depict the vertices of an elementary graph of smooth type using white dots.

Example 4.4. An elementary graph of smooth type. Let $Y_i = \{y = \varphi_i(x)\}$, $1 \leq i \leq 5$, where $\varphi_1(x) = x^2$, $\varphi_2(x) = x^3 + x^4$, $\varphi_3(x) = x^3 - x^4$, $\varphi_4(x) = -x^3 + x^4$ and $\varphi_5(x) = -x^3 - x^4$. Set $Y = \cup_{i=1}^5 Y_i$.

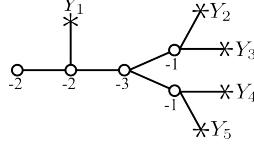


FIGURE 1. The elementary graph \mathcal{R}_Y .

5. GEOMETRY OF CONTINUOUS FRACTIONS

Let (X, o) be a germ of a smooth complex manifold. Let Σ and C be two smooth transversal curves of (X, o) . Choose a system of local coordinates (x, y) of (X, o) such that $\Sigma = \{x = 0\}$ and $C = \{y = 0\}$. Let Ξ be the union of $\{\Sigma, C\}$ with the set of divisors that are created when we desingularize the curves $\{y^k - x^n = 0\}$, with $(k, n) = 1$. We will identify each element of Ξ with each of its strict transforms.

Let \mathbb{Q}_+ be the set of positive integers. Set $\overline{\mathbb{Q}}_+ = \mathbb{Q}_+ \cup \{0, +\infty\}$. The main purpose of this section is to prove Lemma 5.1.

Let us give an invariant definition of Ξ . Set $X_0 = X$, $D_0 = \Sigma \cup C$. Let $\pi_n : X_n \rightarrow X_{n-1}$ be the blow up of X_{n-1} along the singular locus of D_{n-1} . Set $D_n = \pi_n^{-1}(D_{n-1})$. Let \mathcal{R}_n be the dual graph of D_n . We will identify each irreducible component of each D_n with the corresponding strict transforms by blow up. Let Ξ_n be the set of equivalence classes of the vertices of \mathcal{R}_n . Set $\Xi = \cup_n \Xi_n$. If $E, F \in \Xi$ have representatives E', F' such that $E' \cap F' \neq \emptyset$, we will denote by $E * F$ the equivalence class of the divisor created by the blow up of $E' \cap F'$. We will identify an element of Ξ with a convenient representative whenever this identification does not create a serious ambiguity.

Let $E, F, G \in \Xi$. There is m such that E, F, G are vertices of \mathcal{R}_m . We say that F is *between* E and G , and write $E - F - G$, if F is a vertex of each path of \mathcal{R}_m that connects E and G . Since the graphs \mathcal{R}_m are linear, the relation $E < F$ if $\Sigma - E - F$ is a total order of Ξ .

Given integers a_1, \dots, a_n , $a_1 \geq 0$, $a_2, \dots, a_n \geq 1$, $n \geq 1$, we define recursively the rational number $[a_1, \dots, a_n]$ by $[a_1] = a_1$ and $[a_1, \dots, a_n] = a_1 + 1/[a_2, \dots, a_n]$, for $n \geq 2$. Each non negative rational number admits an *expansion in continuous fraction* $[a_1, \dots, a_n]$. Each positive rational number admits two expansions. A positive integer n admits the expansions $[n]$ and $[n - 1, 1]$. If $\alpha \in \mathbb{Q}_+ \setminus \mathbb{Z}$, α admits expansions $[a_1, \dots, a_n]$ and $[a_1, \dots, a_n - 1, 1]$, with $a_n \geq 2$. We set $[n, 0] = +\infty$, for each non negative integer n .

If $a_n \geq 1$, we define the *length* of $[a_1, \dots, a_n]$ as $|[a_1, \dots, a_n]| = a_1 + \dots + a_n$. We set $|0| = |+\infty| = 0$. Given $\alpha = [a_1, \dots, a_k]$, $\beta = [b_1, \dots, b_\ell] \in \overline{\mathbb{Q}}_+$,

we say that $\alpha, \beta \in \overline{\mathbb{Q}}_+$ are *related*, and write $\alpha \sim \beta$, if there is a non negative integer n such that $\alpha = [b_1, \dots, b_\ell, n]$ or $\beta = [a_1, \dots, a_k, n]$. If $\beta = [a_1, \dots, a_k, n]$, we define the *convolution of α and β* as the positive rational $\alpha * \beta = \beta * \alpha = [a_1, \dots, a_k, n+1]$.

$$(7) \quad \text{If } \alpha \sim \beta, \quad \alpha \sim \alpha * \beta \text{ and } |\alpha * \beta| = \max\{|\alpha|, |\beta|\} + 1.$$

Given $\alpha, \beta, \gamma \in \overline{\mathbb{Q}}_+$, we say that β is *between α and γ* , and write $\alpha - \beta - \gamma$, if $\alpha < \beta < \gamma$ or $\gamma < \beta < \alpha$. Given $R \subset \overline{\mathbb{Q}}_+$ and $\alpha, \beta \in R$, we say that α and β are *contiguous* in R , and write $\alpha R \beta$, if there is no $\gamma \in R$ such that $\alpha - \gamma - \beta$.

Notice that $0 * 1/n = [0] * [0, n] = [0, n+1] = 1/(n+1)$, $n * +\infty = [n] * [n, 0] = [n, 1] = n+1$ and

$$0 - \frac{1}{n+1} - \frac{1}{n}, \quad n - n+1 - +\infty$$

Let $\alpha, \beta, \gamma, \delta \in \mathbb{Q}_+$, $\delta = [a_1, \dots, a_k]$, with $k \geq 1$. If $\alpha - \beta - \gamma$,

$$\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma} \text{ and } \left(a_k + \frac{1}{\alpha}\right) - \left(a_k + \frac{1}{\beta}\right) - \left(a_k + \frac{1}{\gamma}\right).$$

Hence $[\delta, \alpha] - [\delta, \beta] - [\delta, \gamma]$. Therefore $\alpha - \alpha * \beta - \beta$ whenever $\alpha \sim \beta$. Moreover, $\alpha \sim \beta$ and $\alpha < \beta$ implies that

$$(8) \quad \alpha < \alpha * \beta < \beta.$$

Relation (8) also holds if $\alpha = 0$ or $\beta = +\infty$.

Let $\alpha, \beta \in \mathbb{Q}_+$. We say that α *precedes* β , and we write $\alpha \prec \beta$, if $\beta = [a_1, \dots, a_k]$ and there are integers ℓ, b such that $\alpha = [a_1, \dots, a_\ell, b]$ and $0 \leq \ell \leq k-1, 1 \leq b \leq a_{\ell+1}$. The partial order \prec induces in \mathbb{Q}_+ a structure of tree with root 1. We say that α *immediately precedes* β if $\alpha \prec \beta$ and $|\beta| = |\alpha| + 1$.

Given two divisors E, F of a resolution graph \mathcal{R} , we will say that E *precedes* F and write $E \prec F$ if there is a sequence of divisors $E = E_0, E_1, \dots, E_n = F$ such that E_i is created blowing up a point of E_{i-1} , $1 \leq i \leq n$.

Lemma 5.1. *There is a bijection $\alpha \mapsto E_\alpha$ from $\overline{\mathbb{Q}}_+$ onto Ξ such that*

- (a) $\alpha < \beta$ if and only if $E_\alpha < E_\beta$.
- (b) If $\alpha \sim \beta$, $E_\alpha * E_\beta = E_{\alpha * \beta}$.
- (c) $\alpha \prec \beta$ if and only if $E_\alpha \prec E_\beta$.

We will denote the inverse of $\alpha \mapsto E_\alpha$ by $E \mapsto \alpha^E$.

Proof. If \mathcal{R}_m has vertices $E^0 < E^1 < \dots < E^{r_m}$, \mathcal{R}_{m+1} has vertices $E^0 < E^0 * E^1 < E^1 < E^1 * E^2 < E^2 < \dots < E^{r_m}$. Since \mathcal{R}_0 has two vertices, \mathcal{R}_m has $2^m + 1$ vertices, for each $m \geq 1$.

Set $\overline{\mathbb{Q}}_m = \{\alpha \in \overline{\mathbb{Q}}_+ : |\alpha| \leq m\}$ for each $m \geq 0$. Each $\alpha \in \mathbb{Q}_+$ immediately precedes two elements of \mathbb{Q}_+ : 1 immediately precedes 2 and $1/2$; if $\alpha = [a_0, \dots, a_k]$ and $a_k \geq 2$, α immediately precedes $[a_0, \dots, a_k + 1]$, $[a_0, \dots, a_k -$

$1, 2]$ and no other rational numbers. Hence $\#\{\alpha : |\alpha| = m\} = 2^{m-1}$ and $\#\overline{\mathbb{Q}}_m = 2^m + 1$, for each $m \geq 1$.

Let us show that, for each $m \geq 1$:

(1) $_m$ $\overline{\mathbb{Q}}_m = \{\alpha_0, \dots, \alpha_{2^m}\}$, $\alpha_0 < \dots < \alpha_{2^m}$, $|\alpha_i| = m$ if i odd, $|\alpha_i| < m$ if i even.

(2) $_m$ If $\alpha, \beta \in \overline{\mathbb{Q}}_m$ and $\alpha \overline{\mathbb{Q}}_m \beta$, $\alpha \sim \beta$.

(3) $_m$ There is a bijection $\alpha \mapsto E_\alpha$ from $\overline{\mathbb{Q}}_m$ onto \mathcal{R}_m such that (a), (c) hold for $\alpha, \beta \in \overline{\mathbb{Q}}_m$ and (b) holds for $\alpha, \beta \in \overline{\mathbb{Q}}_{m-1}$.

Notice that $\overline{\mathbb{Q}}_0 = \{0, +\infty\}$. If $|\alpha| = 1$, $\alpha = 1$. Hence $\overline{\mathbb{Q}}_1 = \{0, 1, +\infty\}$. Since $+\infty = [0, 0]$, $0 \sim +\infty$ and $0 * +\infty = [0, 1] = 1$. By (7), $0 \sim 1$ and $1 \sim +\infty$. If $|\alpha| = 2$, $\alpha = [0, 2] = 1/2$ or $\alpha = [1, 1] = 2$. Since $0 * 1 = [0] * [0, 1] = [0, 2]$ and $1 * +\infty = [1] * [1, 0] = [1, 1] = 2$, $0 \sim 1/2$, $1/2 \sim 1$, $1 \sim 2$ and $2 \sim +\infty$. Assume that (1) $_m$ holds for $i = 1, 2$. By (7) and (8),

$$|\alpha_{2i-2} * \alpha_{2i-1}| = |\alpha_{2i-1} * \alpha_{2i}| = m + 1$$

and

$$(2) \quad \alpha_{2i-2} < \alpha_{2i-2} * \alpha_{2i-1} < \alpha_{2i-1} < \alpha_{2i-1} * \alpha_{2i} < \alpha_{2i},$$

for $1 \leq i \leq 2^{m-1}$. Hence (1) $_{m+1}$ holds and (2) $_{m+1}$ follows from (7).

Set $E_0 = \Sigma$ and $E_{+\infty} = C$. Assume that (3) $_m$ holds. By (b), E has a unique extension to $\overline{\mathbb{Q}}_{m+1}$. By (9), (a) holds for $\alpha, \beta \in \overline{\mathbb{Q}}_{m+1}$. By the definitions of " $\alpha * \beta$ " and " $E * F$ ", (b) holds for $\alpha, \beta \in \overline{\mathbb{Q}}_m$.

Assume $\alpha \in \mathbb{Q}_m$, $|\beta| = m + 1$.

Assume $\alpha \prec \beta$. If $|\alpha| = m$, there is $\gamma \in \mathbb{Q}_m$ such that $\beta = \alpha * \gamma$. Hence $E_\alpha \prec E_\beta$. Otherwise, there is δ such that $|\delta| = m$ and $\alpha \prec \delta \prec \beta$. Hence $E_\alpha \prec E_\delta \prec E_\beta$.

Assume $E_\alpha \prec E_\beta$. If $|\beta| = |\alpha| + 1$, there is $\gamma \in \mathbb{Q}_m$ such that $E_\beta = E_\alpha * E_\gamma$. Hence $\beta = \alpha * \gamma$ and $\alpha \prec \beta$. Otherwise, there are $\gamma, \delta \in \mathbb{Q}_m$ such that $E_\beta = E_\gamma * E_\delta$ and $E_\alpha \prec E_\gamma$. Hence $\alpha \prec \gamma \prec \gamma * \delta = \beta$. \square

6. RESOLUTION GRAPHS OF CUSP TYPE

We keep the notations of Section 4. Let Y be a germ of an irreducible plane curve of (X, o) , $Y \neq \Sigma$. Assume that if Y is singular, Y has maximal contact with C or Y has maximal contact with Σ . Assume Y has at most one Puiseux exponent. Given $\alpha > 1$ [$\alpha < 1, \alpha = 1$] we say that Y is a *cuspidal exponent* α relative to (Σ, C) if $o(Y, C) = \alpha$ [$o(Y, \Sigma) = 1/\alpha, Y$ is transversal to C and Σ]. We assume $\alpha = p/q$, where p, q are positive integers and $(p, q) = 1$.

Let (x, y) be a system of local coordinates of (X, o) such that $\Sigma = \{x = 0\}$ and $C = \{y = 0\}$. If Y is a cuspidal exponent α and $\alpha \geq 1$ [$\alpha < 1$], there is $\varepsilon \in \mathbb{C}\{x^{1/q}\}$ [$\varepsilon \in \mathbb{C}\{y^{1/p}\}$] such that $Y = \{y = x^\alpha \varepsilon\}$ [$Y = \{x = y^\alpha \varepsilon\}$] and $\varepsilon(0) \neq 0$. We say that $\varepsilon(0)$ is the *coefficient of Y relative to the system of local coordinates (x, y)* . We say that C has exponent $+\infty$ and coefficient 0. Notice that the coefficient of a cusp depends on the system of local

coordinates but the fact that two cusps have the same exponent but different coefficients does not.

Lemma 6.1. *Let $Y [Y_i]$ be a cusp of exponent α relatively to (Σ, C) , $i = 1, 2$. Let $\tilde{Y} [\tilde{Y}_i]$ be the strict transform of $Y [Y_i]$ by the minimal resolution of $Y \cup \Sigma \cup C [Y_i \cup \Sigma \cup C]$. Then*

- (a) \tilde{Y} is transversal to E_α .
- (b) $\tilde{Y}_1 \cap \tilde{Y}_2 = \emptyset$ if and only if \tilde{Y}_1 and \tilde{Y}_2 have different coefficients.

Proof. By induction in $|\alpha|$. □

Lemma 6.2. *Let (Y, Σ) be a logarithmic plane curve. Assume \mathcal{R}_Y^Σ linear. Assume that $\tilde{\Sigma}$ has valence 1 and the terminal vertex of \mathcal{R}_Y^Σ does not have weight -1 . Then*

- (a) *Each branch of Y has at most one Puiseux exponent.*
- (b) *There is a smooth curve C transversal to Σ such that all singular branches of Y have maximal contact with C or Σ .*
- (c) *If Y_i, Y_j are smooth branches of Y transversal to Σ , $o(Y_i, Y_j) \leq \tau_C(Y)$.*
- (d) *All branches of Y are cusps relatively to (Σ, C) . If two branches of Y are cusps with the same exponent, they have different coefficients.*

Proof. Statement (a) is well known. See for instance [1]. Let us prove (b). Let λ_i be the exponent of each singular branch Y_i of Y transversal to Σ . If the curve C does not exist, there are i, j such that $o(Y_i, Y_j) < \min\{\lambda_i, \lambda_j\}$. After blowing up $Y_i \cap Y_j$ $o(Y_i, Y_j)$ -times, the curves Y_i and Y_j are still singular and intersect different smooth points of $D_{o(Y_i, Y_j)}$. Hence \mathcal{R}_Y^Σ is not linear. Assume that (c) does not hold. Set $\tau = \tau_C(Y)$. Let $Y' [\Sigma']$ be the germ of $Y^{(\tau)} [E_\tau]$ at $Y_i^{(\tau)} \cap Y_j^{(\tau)}$. By Lemma 6.1, the strict transforms of the singular branches of Y only intersect divisors E such that $\alpha^E < \tau$. Hence, the curve (Y', Σ') verifies the conditions of Theorem 4.2. Therefore $\mathcal{R}_{Y'}^{\Sigma'}$ and \mathcal{R}_Y^Σ have a terminal vertex of weight -1 .

Statement (d) follows from Lemma 6.1 and the linearity of \mathcal{R}_Y^Σ . □

Let (Y, Σ) be a logarithmic plane curve such that \mathcal{R}_Y^Σ verifies the conditions of Lemma 6.2. Assume Y_i is a branch of Y such that $o(Y_i, C) > \tau_C(Y)$. By statement (c), Y_i is smooth and $o(Y_j, C) \leq \tau_C(Y)$, for all $j \neq i$. Hence we can take $C = Y_i$. Under these conditions, the set of divisors of \mathcal{R}_Y^Σ is naturally identified with a subset of Ξ (and a subset of $\overline{\mathbb{Q}}_+$).

Lemma 6.3. *Assume (Y, Σ) verifies the conditions of Lemma 6.2. Then:*

- (a) $E_\alpha \in \mathcal{R}_Y^\Sigma$ if and only if α belongs to the smallest subtree R of $(\mathbb{Q}_+, <)$ that contains 1 and the exponents of the branches of Y .
- (b) If $\alpha = [a_1, \dots, a_k] \in R$,

$$\omega_{E_\alpha} = -1 - \max\{c, 0 : [a_1, \dots, a_k, c] \in R\} - \max\{c, 0 : [a_1, \dots, a_k - 1, 1, c] \in R\}.$$

Proof. Let $Y [Z]$ be a cusp of type $[a_1, \dots, a_k, c] [[a_1, \dots, a_k - 1, 1, d]]$. After blowing up $D_\ell \cap Y^{(\ell)}$, $\ell = 0, 1, \dots, |\alpha| - 1$, one creates the divisor E_α . Moreover, the cusps $Y^{(|\alpha|)}, Z^{(|\alpha|)}$ intersect E_α at different points, $o(Y^{(|\alpha|)}, E_\alpha) = c$ and $o(Z^{(|\alpha|)}, E_\alpha) = d$.

After blowing up $D_\ell \cap Y^{(\ell)}$, $\ell = |\alpha|, \dots, |\alpha| + c - 1$ and $D_\ell \cap Z^{(\ell)}$, $\ell = |\alpha|, \dots, |\alpha| + d - 1$, $Y^{(|\alpha|+c)}$ and $Z^{(|\alpha|+d)}$ no longer intersect E_α . Moreover, E_α has weight $-1 - c - d$. \square

Let \mathcal{R}^Σ be a logarithmic resolution graph verifying the conditions of Lemma 6.2. We can give an explicit embedding of the set of vertices of \mathcal{R}^Σ into $\mathbb{Q}_+ \cup \{0\}$ by setting $\alpha^\Sigma = 0$, $\alpha^C = +\infty$ and $\alpha^E = \alpha^{E'} * \alpha^{E''}$ whenever we are at a resolution step such that $\omega_E = -1$, $E' \cap E, E'' \cap E \neq \emptyset$. Let $\tau(\mathcal{R}^\Sigma)$ be the smallest positive integer τ such that for each non divisor E of \mathcal{R}^Σ with $\alpha^E \notin \mathbb{Z}$, $\alpha^E < \tau$.

Assume that the resolution graph \mathcal{R}_Y of Y is linear, its root has valence 1 and its vertex does not have weight -1 . Let Σ be a smooth curve transversal to Y . We obtain \mathcal{R}_Y^Σ from \mathcal{R}_Y connecting Σ to the root of \mathcal{R}_Y and setting $\omega_\Sigma = -2$. By Lemma 6.2, the set of divisors of \mathcal{R}_Y is naturally identified with a subset of \mathbb{Q}_+ . Set $\tau(\mathcal{R}_Y) = \tau(\mathcal{R}_Y^\Sigma)$.

Definition 6.4. Let $\mathcal{R} [\mathcal{R}^\Sigma]$ be a [logarithmic] resolution graph. We say that $\mathcal{R} [\mathcal{R}^\Sigma]$ is a [log]elementary graph of cusp type if:

- (a) $\mathcal{R} [\mathcal{R}^\Sigma]$ is a linear graph and Σ has valence 1.
- (b) $\mathcal{R} [\mathcal{R}^\Sigma]$ is not a [log] elementary graph of smooth type.
- (c) If an integer divisor E of $\mathcal{R} [\mathcal{R}^\Sigma]$ intersects a strict transform of a branch of the logarithmic curve, $\alpha^E = \tau(\mathcal{R})$ [$\alpha^E = \tau(\mathcal{R}^\Sigma)$].

Condition (c) guarantees the uniqueness of the decomposition (see Definition 7.1 and Examples 7.6, 7.7).

Theorem 6.5. Let (Y, Σ) be a logarithmic plane curve. Its resolution graph is a log elementary graph of cusp type if and only if:

- (a) there is a smooth curve C transversal to Σ and a system of local coordinates (x, y) such that $\Sigma = \{x = 0\}$, $C = \{y = 0\}$ and all branches of Y are cusps relatively to (Σ, C) .
- (b) If two branches of Y have the same exponent, they have different coefficients.
- (c) If Y_i is a branch with integer exponent α , $\alpha = \tau_C(Y)$.
- (d) Y has a singular branch or a branch tangent to Σ .

Proof. Assume \mathcal{R}_Y^Σ is a log elementary graph of cusp type. Statement (d) follows from condition (b) (of Definition 6.4) and Theorem 4.2. Statements (a), (b) follow from condition (a) and Lemma 6.2. Assume that Y has several smooth branches transversal to Σ . By Lemma 6.2 its exponents are smaller than or equal to $\tau_C(Y)$. By condition (c), these exponents are equal to $\tau_C(Y)$.

Let us prove the converse. Condition (a) follows from statements (a), (b) and Lemmas 5.1, 6.1. Condition (b) follows from statement (d) and Lemma 6.3. Condition (c) follows from statement (c). \square

We will depict the vertices of an elementary graph of cusp type using black dots.

Example 6.6. An elementary graph of cusp type. Let $Y_i = \{y = \varphi_i(x)\}$, $1 \leq i \leq 3$, where $\varphi_1(x) = x^{3/2}$, $\varphi_2(x) = x^{5/3}$ and $\varphi_3(x) = x^{5/2}$. Set $Y = \cup_{i=1}^3 Y_i$.

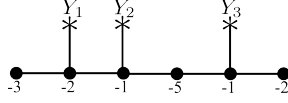


FIGURE 2. The elementary graph \mathcal{R}_Y .

Corollary 6.7. Let Y be the germ of a plane curve. Its resolution graph is a elementary graph of cusp type if and only if:

- (a) There is a smooth curve Σ transversal to Y and there is a smooth curve C transversal to Σ such that all branches of Y are cusps relatively to (Σ, C) .
- (b) If two branches of Y have the same exponent, they have different coefficients.
- (c) If Y_i is a branch with integer exponent α , $\alpha = \tau_C(Y)$.
- (d) Y has a singular branch.

7. DECOMPOSITION OF A RESOLUTION GRAPH

Definition 7.1. Let \mathcal{G} be an elementary graph, E a vertex of \mathcal{G} and Z a branch that intersects E . Let \mathcal{H} be a log elementary graph with root Σ . We glue \mathcal{G} with \mathcal{H} at Z withdrawing Z , identifying Σ with E and replacing ω_E by $\omega_E + \omega_\Sigma + 1$. The gluing of \mathcal{G} with \mathcal{H} at Z is admissible if E is not a black root and one of the following conditions is verified:

- (a) E and Σ have different colors.
- (b) E is black and the number of divisors connected to E plus the number of branches connected to E is greater than 2.

Let Λ be a rooted tree with root ϕ . Let $(\mathcal{D}_u)_{u \in \Lambda}$ be a family of elementary graphs. Assume all log elementary with the possible exception of \mathcal{D}_ϕ . Associate to each vertex u of Λ , $u \neq \phi$, its father $f(u)$, a divisor E_u of $\mathcal{D}_{f(u)}$ and a branch Y_u that intersects E_u . Assume that the map $u \mapsto Y_u$ is injective. We call gluing of $(\mathcal{D}_u)_{u \in \Lambda}$ to the graph we obtain gluing $\mathcal{D}_{f(u)}$ with \mathcal{D}_u at Y_u for each $u \in \Lambda \setminus \{\phi\}$.

We say that a resolution graph admits a decomposition into elementary graphs if it is a gluing of elementary graphs. A decomposition into elementary graphs of a resolution graph is called admissible if, the gluing of $\mathcal{D}_{f(u)}$ with \mathcal{D}_u at Y_u is admissible for each $u \in \Lambda \setminus \{\phi\}$.

Theorem 7.2. *A gluing of elementary graphs is a resolution graph.*

Proof. Let us prove the theorem by induction on the number of vertices of Λ . The theorem holds if $\Lambda = \{\phi\}$. Let u be a terminal vertex of Λ , $u \neq \phi$. Set $\Lambda_0 = \Lambda \setminus \{u\}$ and let o be the intersection point of Y_u and E_u . By the induction hypothesis, the gluing of $(\mathcal{D}_v)_{v \in \Lambda_0}$ is the resolution graph of a plane curve H . Let $\pi : \tilde{X} \rightarrow (X, \sigma)$ be the resolution of H . Set $D = \pi^{-1}(\sigma)$. Let \tilde{H} be the strict transform of H by π . Let \tilde{H}_u be the germ \tilde{H} at o . By Theorems 6.5, 4.2 there is a germ of a logarithmic curve (W, Σ) with resolution graph \mathcal{D}_u . We can assume that Σ is the germ of E_u at o . Set $Y = \pi((\tilde{H} \setminus H_u) \cup W)$. The resolution graph of Y is obtained gluing the resolution graph of H and the resolution graph of (W, Σ) through the identification of Σ with a germ of E_u . Let \tilde{E} be the strict transform of E_u by the sequence of blow ups that desingularizes $(\tilde{H} \setminus H_u) \cup W$. If κ denotes the number of blow ups necessary to separate W and Σ , $\omega_{\tilde{E}}$ equals $\omega_{E_u} - \kappa$. \square

We call *leading term* of a Puiseux series $\sum_{\alpha} a_{\alpha} x^{\alpha}$ to $a_{\beta} x^{\beta}$, where β is the smallest positive rational $\gamma \in \mathbb{Q}$ such that $a_{\gamma} \neq 0$. If $Z = \{y = \sum_{\alpha} a_{\alpha} x^{\alpha}\}$ is a germ of plane curve, set $Z^{[\ell]} = \{y = \sum_{\alpha \leq \ell} a_{\alpha} x^{\alpha}\}$.

Lemma 7.3. *Let (Y, Σ) be a logarithmic plane curve, with Σ smooth. There is a curve Y^{ϕ} such that:*

- (a) *the Puiseux expansion of each irreducible component Y_i^{ϕ} of Y^{ϕ} is a truncation of the Puiseux expansion of some irreducible component Y_i of Y ;*
- (b) *the resolution graph of (Y^{ϕ}, Σ) is a log elementary graph;*
- (c) *the sequence of blow ups that desingularize (Y, Σ) and (Y^{ϕ}, Σ) coincide up to the step when the inverse image of $Y^{\phi} \cup \Sigma$ is normal crossings;*
- (d) *let $\pi_{\phi} : X^{\phi} \rightarrow X$ [$\pi : \tilde{X} \rightarrow X$] be the sequence of blow ups that desingularizes (Y^{ϕ}, Σ) [(Y, Σ)]. Let \tilde{Y} be the strict transform of Y by π_{ϕ} . Set $D^{\phi} = \pi_{\phi}^{-1}(\Sigma \cap Y)$. Let σ be a point of D^{ϕ} such that $D^{\phi} \cup \tilde{Y}$ is not normal crossings at σ . Let $D_{\sigma}^{\phi}[\tilde{Y}_{\sigma}]$ be the germ of $D^{\phi}[\tilde{Y}]$ at σ . Then there is a truncation $\tilde{Y}_{\sigma}^{\phi}$ of \tilde{Y}_{σ} such that the resolution graph of $(\tilde{Y}_{\sigma}^{\phi}, D_{\sigma}^{\phi})$ is an elementary graph and the gluing of $\mathcal{R}_{Y^{\phi}}^{\Sigma}$ with $\mathcal{R}_{\tilde{Y}_{\sigma}^{\phi}}^{D_{\sigma}^{\phi}}$ at $\tilde{Y}_{\sigma}^{\phi}$ is admissible.*

Proof. Set $J = \{i \in I : Y_i \text{ is transversal to } \Sigma\}$ and $K = I \setminus J$. For $i \in J$, set λ_i as the first Puiseux exponent of Y_i , if Y_i is singular, and $\lambda_i = +\infty$ otherwise. If Y_i is smooth for each $i \in J$, let (x, y) be a system of local coordinates such that $\Sigma = \{x = 0\}$. Assume that there is $i \in J$ such that Y_i is singular. Let ρ be the supremum of the set of integers ℓ such that there is a smooth curve C that verifies, for each $i \in J$, $o(Y_i, C) \geq \ell$ or $o(Y_i, C) = \lambda_i$. Choose a smooth curve C such that C is transversal to Σ and, for each $i \in J$, $o(Y_i, C) \geq \rho$ or $o(Y_i, C) = \lambda_i$. Let (x, y) be a system of local coordinates

such that $\Sigma = \{x = 0\}$ and $C = \{y = 0\}$. Let τ_0 be the supremum of λ_i such that $i \in J$, Y_i is singular and $\lambda_i \leq \rho$. Let τ be the smallest positive integer bigger than τ_0 .

(A) Assume $K = \emptyset$ and $\tau = 1$. Let q_i be the supremum of the set of integers ℓ such that $Y_i^{[\ell]}$ is smooth and $o(Y_i^{[\ell]}, Y_j^{[\ell]}) \in \mathbb{Z} \cup \{+\infty\}$ for each $j \in I$. Set $Y_i^\phi = Y_i^{[q_i]}$ for each $i \in I$. Set $Y^\phi = \cup_{i \in I} Y_i^\phi$.

(B) Assume $K \neq \emptyset$ or $\tau \geq 2$. If $i \in K$ and $Y_i = \{x = \psi_i(y)\}$, set $Y_i^\phi = \{x = \tilde{\psi}_i(y)\}$, where $\tilde{\psi}_i$ is the leading term of ψ_i . Assume $i \in J$ and $Y_i = \{y = \varphi_i(x)\}$. If $o(Y_i, C) > \tau$, set $Y_i^\phi = C$. Otherwise, set $Y_i^\phi = \{y = \tilde{\varphi}_i(x)\}$, where $\tilde{\varphi}_i$ is the leading term of φ_i . Set $Y^\phi = \cup_{i \in I} Y_i^\phi$.

By construction, (Y^ϕ, Σ) verifies condition (a). In case (A) [(B)], (Y^ϕ, Σ) verifies the conditions of Theorem 4.2 [Theorem 6.5], hence statement (b) holds.

Statement (c) follows from Theorems 2.1 and 2.3.

Let us show that statement (d) holds.

Assume (A). Let $\tilde{Y}_i [Y_i^\phi]$ be the strict transform of $Y_i [Y_i^\phi]$ by π_ϕ . Choose σ such that $D^\phi \cap \tilde{Y}$ is not normal crossings at σ . By the definition of the q_i 's, there is $i \in I$ such that $\sigma \in \tilde{Y}_i$, $Y_i^{[q_i+1]}$ is singular and \tilde{Y}_i is either singular or tangent to D^ϕ at σ . Hence \tilde{Y}_σ^ϕ verifies condition (B).

Assume (B). Let \tilde{C} be the strict transform of C by π_ϕ . By the definition of admissibility, we only need to check what happens at the point σ such that $\tilde{C} \cap D^\phi = \{\sigma\}$. Set $J_\phi = \{i \in J : Y_i^\phi = C\}$, $Y_\phi = \cup_{i \in J_\phi} Y_i$. Notice that the germ at σ of the strict transform of Y by π_ϕ equals the strict transform of Y_ϕ by π_ϕ . If $J_\phi = \emptyset$ or Y_ϕ is smooth, nothing is glued to $\mathcal{R}_{Y_\phi}^\Sigma$ at σ and there is nothing to prove. Assume Y_ϕ is singular. By the construction of Y^ϕ , in particular the definitions of ρ and τ , the minimum of the set $\{o(Y_i, Y_j) : i, j \in J_\phi\}$ is an integer. Hence \tilde{Y}_σ^ϕ verifies condition (A). \square

Theorem 7.4. *The resolution graph of a logarithmic plane curve (Y, Σ) , with Σ smooth, admits one and only one admissible decomposition into elementary graphs.*

Proof. Let g_i be the number of Puiseux pairs of the branch Y_i of Y , $i \in I$. Set $h_i = g_i$ if Y_i is transversal to Σ . Otherwise, set $h_i = g_i + 1/2$. Set $h_Y = \max\{h_i : i \in I\}$. Let $\#_Y$ be the cardinality of I . Let us construct an admissible decomposition into elementary graphs $(\mathcal{D}_v)_{v \in \Lambda}$ of \mathcal{R}_Y^Σ by induction on h_Y and $\#_Y$.

Let Y^ϕ be the curve constructed in Lemma 7.3. We will follow the notations of this Lemma.

If $\sigma \in D^\phi$ and $D^\phi \cup \tilde{Y}$ is not normal crossings at σ , $h_{\tilde{Y}_\sigma} < h_Y$ or $\#_{\tilde{Y}_\sigma} < \#_Y$. Hence $(\tilde{Y}_\sigma, D_\sigma^\phi)$ admits an admissible decomposition into log elementary

graphs. Since \mathcal{R}_Y^Σ is the gluing of $\mathcal{R}_{Y^\phi}^\Sigma$ and the graphs $\mathcal{R}_{Y_\sigma^\phi}^{D_\sigma^\phi}$, we obtain an admissible decomposition into elementary graphs of \mathcal{R}_Y^Σ .

Let $(\mathcal{E}_v)_{v \in \Lambda'}$ be another admissible decomposition of \mathcal{R}_Y^Σ . Let us show by induction on h_Y and $\#_Y$ that the two decompositions are equal. It is enough to show that $\mathcal{D}_\phi = \mathcal{E}_\phi$.

Assume that \mathcal{D}_ϕ is of cusp type and \mathcal{E}_ϕ is of smooth type. Let $(E, \Sigma) [(D, \Sigma)]$ be a logarithmic plane curve such that $\mathcal{R}_E^\Sigma = \mathcal{E}_\phi [\mathcal{R}_D^\Sigma = \mathcal{D}_\phi]$. Let $\pi_E : X^E \rightarrow X [\pi_D : X^D \rightarrow X]$ be the sequence of blow ups that desingularizes $(E, \Sigma) [(D, \Sigma)]$. There is a subtree R of (\mathbb{Q}_+, \prec) such that we can identify the set of vertices of \mathcal{D}_ϕ with $\{E_\alpha : \alpha \in R\}$. Let m be the biggest integer of R . By condition (c) of Definition 6.4, the vertices E_1, \dots, E_m of \mathcal{D}_ϕ are also vertices of \mathcal{E}_ϕ and the strict transform of E by π_D intersects E_m at a smooth point of $\pi_D^{-1}(\Sigma)$. Hence, The strict transform of D by π_E intersects E_m at a singular point of $\pi_E^{-1}(\Sigma)$. Therefore the decomposition $(\mathcal{E}_v)_{v \in \Lambda'}$ is not admissible and \mathcal{D}_ϕ and \mathcal{E}_ϕ must be of the same type. Let G be a vertex of \mathcal{E}_ϕ and F a vertex of \mathcal{R}_Y^Σ . Since \mathcal{E}_ϕ is a resolution graph,

$$(10) \quad \text{if } F \prec G, F \text{ is a vertex of } \mathcal{E}_\phi.$$

Assume that there is a vertex of \mathcal{D}_ϕ that is not a vertex of \mathcal{E}_ϕ . Let γ be the biggest element of R such that E_γ is a vertex of \mathcal{E}_ϕ . Let n be the smallest integer bigger than or equal to γ . Then $\gamma = n$. Otherwise $E_n \prec E_\gamma$ and \mathcal{E}_ϕ would not be a resolution graph by (10). Since \mathcal{D}_ϕ is linear and \mathcal{D}_ϕ properly contains \mathcal{E}_ϕ , E has a smooth branch E' such that the strict transform of E' by π_E intersects E_n . Therefore (D, Σ) does not verify condition (c) of Definition 6.4. Hence the decomposition $(\mathcal{E}_v)_{v \in \Lambda'}$ is not admissible.

If there is a vertex E of \mathcal{E}_ϕ that is not a vertex of \mathcal{D}_ϕ , the same argument shows that $(\mathcal{D}_v)_{v \in \Lambda}$ is not admissible.

Assume that \mathcal{D}_ϕ is of smooth type. Hence \mathcal{E}_ϕ is also of smooth type. Assume $\mathcal{D}_\phi \neq \mathcal{E}_\phi$. Hence we can assume that there is a vertex E of \mathcal{D}_ϕ and \mathcal{E}_ϕ is connected to a vertex F of \mathcal{E}_ϕ that is not a vertex of \mathcal{D}_ϕ . Let \mathcal{F} be the subtree of \mathcal{E}_ϕ with vertices the divisors D of \mathcal{E}_ϕ such that $D = F$ or $F \prec D$. Let us reset ω_E in such a way that ω_E plus the set of vertices of \mathcal{F} connected to E equal -1 . We obtain in this way a resolution graph (\mathcal{F}, E) . Since \mathcal{E}_ϕ is of smooth type, \mathcal{F} is of smooth type. Hence the gluing of $(\mathcal{D}_v)_{v \in \Lambda}$ is not admissible. \square

Corollary 7.5. *The resolution graph of a plane curve admits one and only one admissible decomposition into elementary graphs.*

Example 7.6. Let $Y_i = \{y = \varphi_i(x)\}$, $i = 1, 2$, where $\varphi_1(x) = x^2 + x^{7/2}$ and $\varphi_2(x) = x^{7/2}$. We have $Y_i^\phi = \{y = \varphi_i^\phi(x)\}$, $i = 1, 2$, with $\varphi_1^\phi(x) = x^2$ and $\varphi_2^\phi(x) = 0$. Set $Y = Y_1 \cup Y_2$. See Figure 3. The decomposition (C) is not admissible since the resolution graph of $Y_1 \cup Y_2^\phi$ does not verify condition (c) of Definition 6.4.

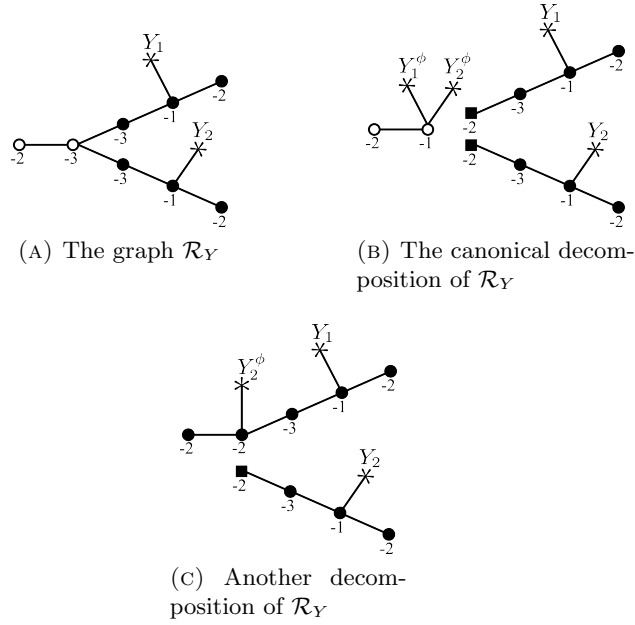


FIGURE 3. Example 7.6.

Example 7.7. Let $Y_i = \{y = \varphi_i(x)\}$, $1 \leq i \leq 4$, where $\varphi_1(x) = x^{3/2}$, $\varphi_2(x) = x^{5/2}$, $\varphi_3(x) = x^2 + x^4$ and $\varphi_4(x) = x^2 - x^4$. We have $Y_i^\phi = \{y = \varphi_i^\phi(x)\}$, $1 \leq i \leq 4$, with $\varphi_1^\phi(x) = \varphi_1(x) = x^{3/2}$, $\varphi_2^\phi(x) = 0$ and $\varphi_3^\phi(x) = \varphi_4^\phi(x) = x^2$. Set $Y = \cup_{i=1}^4 Y_i$. See Figure 4. The decomposition (C) is not admissible since the resolution graph of $Y_1 \cup Y_3^\phi \cup Y_2$ does not verify condition (c) of Definition 6.4.

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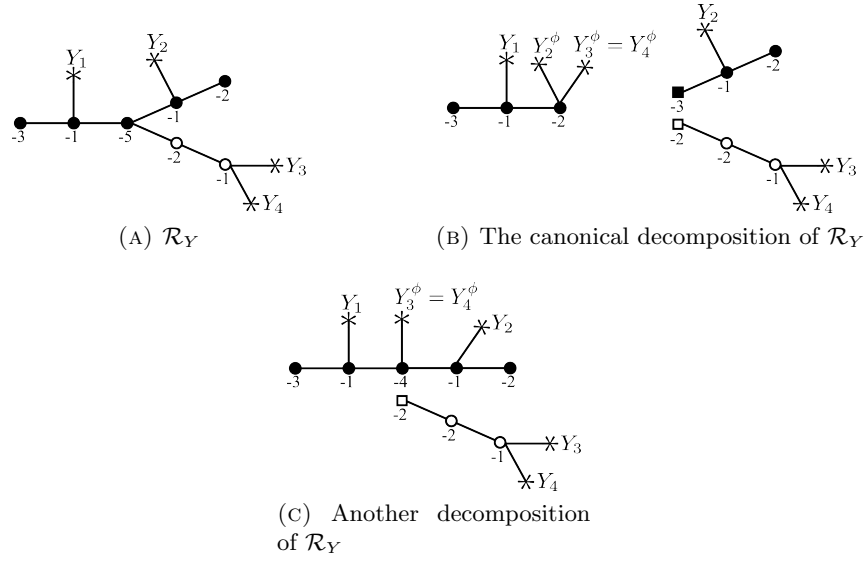


FIGURE 4. Example 7.7.

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This research was partially supported by FEDER and FCT-Plurianual 2011.